Lesson 1.10 … Continuity and Differentiability

WHEN IS A FUNCTION CONTINUOUS ???

Remember that by definition, a function is **continuous** is there are no breaks or interruptions in its graph.

The three-step procedure for testing for continuity at a point $x = c$ must also be satisfied:

A function f is continuous at $x = c$ if and only if …

1. $f(c)$ exists

2. $\lim_{x \to c} f(x)$ exists (here you might have to consider both the left and right-hand limits!)

3. $f(c) = \lim_{x \to c} f(x)$

WHEN IS A FUNCTION DIFFERENTIABLE ???

A function will be **differentiable** if you can find the derivative, the slope of the line tangent to the curve, at every point on the graph.

If we are concerned with a function being differentiable at a specified point, it must be true that the derivative from the left of the x-value in question must equal the derivative from the right of the x-value. Using the 2nd Limit Definition of a Derivative, this means that

$$\underbrace{\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}}_{\substack{\text{Derivative} \\ \text{from the Left} \\ f'_-(x)}} = \underbrace{\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}}_{\substack{\text{Derivative} \\ \text{from the Right} \\ f'_+(x)}}$$

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So what is the relationship between continuity and differentiability?

Think about the Test for Continuity and its three steps.

We stated above that in order for a function to be differentiable at a point, the limit from the left must equal the limit from the right (step 2 of the Test for Continuity).

It follows that if a function is NOT continuous at a given point, then, that it cannot be differentiable at that point either.

This leads us to a BIG THEOREM … that has no actual name. ☹

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Or more simply … differentiability implies continuity.

- The contrapositive of this is also true: "If f is not continuous at $x = c$, then f is not differentiable at $x = c$".
- The converse is NOT necessarily true! (The converse would state that "If f is continuous at c, then f is differentiable at c".)
- Nor is the inverse necessarily true! (The inverse would state that "If f is not differentiable at c, then f is not continuous at c".)

Let us look at an example that shows that a function can be continuous but not differentiable.

Consider the continuous function $f(x) = |x|$.

Written as a piece function, this becomes $f(x) = \begin{cases} -x & \text{when } x \le 0 \\ x & \text{when } x > 0 \end{cases}$

The derivative of this, then, would be $f'(x) = \begin{cases} -1 & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$

$f'(0)$ DNE because the derivative from the left does not equal the derivative from the right: $f'_{-}(0) \ne f'_{+}(0)$. So even though $f(x) = |x|$ is continuous at x = 0, it is NOT differentiable there!

Work through the example problems that have been recorded for you and posted in Canvas, then try the problems below. The solutions to these can be found at the end of this document.

#1) Consider $f(x)=\begin{cases} 2x^2+x-1 & \text{when } x\le 1 \\ 5x-3 & \text{when } x>1 \end{cases}$. Is $f(x)$ continuous and differentiable at $x=1$?

#2) Consider $f(x)=\begin{cases} x^2+2 & \text{when } x\le 1 \\ 2+x & \text{when } x>1 \end{cases}$. Is $f(x)$ continuous and differentiable at $x=1$?

#3) Find the values for a and c that make the piece function differentiable for all Real numbers.

$$f(x)=\begin{cases} ax^2+10 & \text{when } x<2 \\ x^2-6x+c & \text{when } x\ge 2 \end{cases}$$

(Practice problems courtesy of Denise Fuji McLeary for http://www.zendog.org/homework)

Problem Answers:

#1) Let us first apply the Tests for Continuity to determine if this function is continuous at $x = 1$.

1. $f(1) = 2$ so the function value does exist at $x = 1$.
2. Does $\lim_{x\to 1} f(x)$ exist? Yes, because $\lim_{x\to 1^-} f(x) = 2(1)+1-1 = 2$ and $\lim_{x\to 1^+} f(x) = 5(1)-3 = 2$.
3. $f(1) = 2 = \lim_{x\to 1} f(x)$

Therefore, $f(x)$ is continuous at $x = 1$. But is it differentiable???

$f'(x) = \begin{cases} 4x+1 & \text{when } x \le 1 \\ 5 & \text{when } x > 1 \end{cases}$. $f'_-(1) = 4(1)+1 = 5$ and $f'_+(1) = 5$.

Therefore, $f(x)$ is differentiable at $x = 1$.

#2) Let us first apply the Tests for Continuity to determine if this function is continuous at $x = 1$.

1. $f(1) = 3$ so the function value does exist at $x = 1$.
2. Does $\lim_{x\to 1} f(x)$ exist? Yes, because $\lim_{x\to 1^-} f(x) = 1^2 + 2 = 3$ and $\lim_{x\to 1^+} f(x) = 2+1 = 3$.
3. $f(1) = 3 = \lim_{x\to 1} f(x)$

Therefore, $f(x)$ is continuous at $x = 1$. But is it differentiable???

$f'(x) = \begin{cases} 2x & \text{when } x \le 1 \\ 1 & \text{when } x > 1 \end{cases}$. $f'_-(1) = 2(1) = 2$ but $f'_+(1) = 1$.

Therefore, $f(x)$ is NOT differentiable at $x = 1$.

#3) Start by finding the derivative $f'(x) = \begin{cases} 2ax & \text{when } x < 2 \\ 2x-6 & \text{when } x \ge 2 \end{cases}$. Since we want this function to be differentiable for all Real numbers, we need for $f'_-(2) = f'_+(2)$ because if there is going to be a "problem" anywhere with this function, it is going to be at the "split point" between the two pieces.

$$f'_-(2) = f'_+(2)$$
$$2ax = 2x-6$$
$$2a(2) = 2(2)-6 \quad \text{for } x = 2$$
$$\text{So } a = \frac{-1}{2}$$

But if f is known now to be differentiable at $x = 2$, it must therefore also be continuous at $x = 2$. If this is the case, the two pieces must meet up at $x = 2$, meaning that $ax^2 + 10 = x^2 - 6x + c$ at $x = 2$.

$$ax^2 + 10 = x^2 - 6x + c$$
$$\frac{-1}{2}(4) + 10 = 4 - 12 + c \quad \text{for } x = 2$$
$$\text{So } c = 16$$