Lesson 2.5 … Derivatives and Integrals with Polar Curves

INTRODUCTION

In this lesson, we will investigate how derivatives and integrals of polar functions can help describe the behavior of those curves.

Recall earlier in this unit when we analyzed derivatives and integrals of parametrically-defined curves to provide us with information about the behavior of those curves. We'll do the same in this lesson, only this time with polar curves.

In thinking about the slope of a polar curve, we find again that, though the polar curves are described in terms of r and θ, we still want to calculate $\frac{dy}{dx}$ – ***not*** $\boldsymbol{\frac{dr}{d\theta}}$. How do we find $\frac{dy}{dx}$ when we are analyzing polar curves?

DERIVATIVES AND POLAR EQUATIONS

See the video posted for you in Canvas on the derivation of the derivative $\frac{dy}{dx}$ of a polar equation.

Example #1: Consider the graph of the polar curve $r = 2 + 4sin\theta$. What is the equation of the tangent line to the curve when $\theta = \pi$?

Recall from the video that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(rsin\theta)}{\frac{d}{d\theta}(rcos\theta)} = \frac{\frac{dr}{d\theta}(sin\theta) + rcos\theta}{\frac{dr}{d\theta}(cos\theta) - rsin\theta}$$

Using this, we find the following: $\frac{dr}{d\theta} = 4cos\theta$

So,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}(sin\theta) + rcos\theta}{\frac{dr}{d\theta}(cos\theta) - rsin\theta} = \frac{(2 + 4sin\theta)cos\theta + sin\theta(4cos\theta)}{(2 + 4sin\theta)(-sin\theta) + cos\theta(4cos\theta)}$$

Evaluating this general equation for our slope at $\theta = \pi$, we find that $\frac{dy}{dx}|_{\theta=\pi} = \frac{-2+0}{0+4} = -\frac{1}{2}$

When $\theta = \pi$, $(r, \theta) = (2, \pi)$ and $(x, y) = (-2, 0)$. So, in rectangular coordinates, the equation of the tangent line is $\boldsymbol{y = -\frac{1}{2}(x+2) = -\frac{1}{2}x - 1}$.

Now, if $y = mx + b$, then $r sin\theta = mr cos\theta + b$. So, $r(sin\theta - mcos\theta) = b$. Therefore, $r = \frac{b}{sin\theta - mcos\theta}$. So, the equation of the tangent line in polar coordinates is

$$\boldsymbol{r = -\frac{1}{sin\theta + \frac{1}{2}cos\theta}}$$

Now try these! Answers are included at the end of this document.

#1) For the cardioid $r = 1 + \sin\Theta$, find the slope of the tangent line when $\Theta = \pi/3$.

#2) Find the horizontal and vertical tangent lines of $r = \sin\Theta$ for $\Theta \in [0, \pi]$.

ARC LENGTH OF A POLAR CURVE

If no segment of the polar curve $r = f(\Theta)$ is traced more than once as Θ increases from α to β, and if $\frac{dr}{d\Theta}$ is continuous for $\alpha \le \Theta \le \beta$, then the ***arc length*** L from $\Theta = \alpha$ to $\Theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\Theta}\right)^2}\, d\Theta$$

For an explanation as to how this formula is derived, please visit https://www.khanacademy.org/math/integral-calculus/solid_revolution_topic/arc-length-polar-graphs/v/polar-arc-length-formula.

Now try these! You are welcome to use your graphing calculator to evaluate the integral you set up. Answers are included at the end of this document.

#3) Calculate the arc length of the entire circle $r = 3$. Use the arc length formula above but then think of another way to find the area.

#4) Calculate the arc length of the entire cardioid $r = 2 - 2\cos\Theta$.

AREAS OF POLAR CURVES

Just as we saw with functions defined in rectangular coordinates, we can use the Calculus to find areas enclosed by polar curves!

See the next two videos posted for you in Canvas. The first walks you through the derivation of the formula to calculate area enclosed by a polar curve. The other provides two worked examples of finding the area enclosed by a polar curve.

When you are done viewing this second video with the worked examples, please do work through Examples 6-9 on Pages 724-725 of our Anton textbook.

Problem Answers:

#1) Applying the formula for the derivative of a polar curve, we get that $\frac{dy}{dx}=\frac{(\cos\Theta)(\sin\Theta)+(1+\sin\Theta)(\cos\Theta)}{(\cos\Theta)(\cos\Theta)-(1+\sin\Theta)(\sin\Theta)}=\frac{\cos\Theta+2\sin\Theta\cos\Theta}{\cos^2\Theta-\sin\Theta-\sin^2\Theta}=\frac{\cos\Theta+\sin 2\Theta}{\cos 2\Theta-\sin\Theta}$ by application of the Double-Angle Identities. Then $\left.\frac{dy}{dx}\right|_{\Theta=\frac{\pi}{3}}=\frac{\frac{1}{2}+\frac{\sqrt{3}}{2}}{\frac{-1}{2}-\frac{\sqrt{3}}{2}}=\frac{1+\sqrt{3}}{-1-\sqrt{3}}=-1$.

#2) Horizontal tangents will be located where $\frac{dy}{d\Theta}=0$ and $\frac{dx}{d\Theta}\neq 0$. Remember that $y=r\sin\Theta$. Therefore, $\frac{dy}{d\Theta}=r(\cos\Theta)+(\sin\Theta)\frac{dr}{d\Theta}=(\sin\Theta)(\cos\Theta)+(\sin\Theta)(\cos\Theta)=\sin 2\Theta$ by the Double-Angle Identity. Setting this equal to 0, we obtain that $\Theta = 0$ or $\Theta = \pi/2$.
Vertical tangents will be located where $\frac{dx}{d\Theta}=0$ and $\frac{dy}{d\Theta}\neq 0$. Remember that $x=r\cos\Theta$. Therefore, $\frac{dx}{d\Theta}=r(-\sin\Theta)+(\cos\Theta)\frac{dr}{d\Theta}=-\sin^2\Theta+\cos^2\Theta=\cos 2\Theta$ by the Double-Angle Identity. Setting this equal to 0, we obtain that $\Theta = \pi/4$ or $\Theta = 3\pi/4$.

#3) $L=\int_0^{2\pi}\sqrt{9+0^2}\,d\Theta=\left[\sqrt{9}\cdot\Theta\right]_0^{2\pi}=6\pi$. OR, just think of equation for circumference of a circle, $C=2\pi\cdot r=2\pi\cdot(3)=6\pi$.

#4) $\frac{dr}{d\Theta}=0+2\sin\Theta$. Then $L=\int_0^{2\pi}\sqrt{(2-2\cos\Theta)^2+(2\sin\Theta)^2}\,d\Theta=16$.